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Pulse Formation and Spectral Shaping in Chaotic Systems

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The control of chaos using tiny perturbations for the formation of time-limited waveforms (pulses) and the synthesis of desired spectral characteristics is discussed. A technique for doing this using a symbolic dynamical system model is developed. This report is intended to demonstrate that the highly successful methods for controlling chaos with microscopic control signals might be used to form pulses and to control the power spectrum of a signal source such as a microwave power source or a laser. Since the control circuitry in a physical system would need to deliver only tiny low-power control perturbations, the realization of this technique in a high-power source would allow for the formation of shaped pulses with the source running in a highly efficient nonlinear chaotic mode. Shaping the power spectrum is of interest for the practical problem of delivering the energy to a band-limited transmission line or to an antenna.

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1. Introduction

Many electrical (and for that matter nonelectrical) devices, including microwave sources [1], lasers [2], solid-state microwave oscillators, and electronic circuits [3], exhibit chaos when pushed to high power levels. The presence of chaos in these devices can limit their efficiency and usefulness.

Recently, several experimental groups have demonstrated that it is possible to control chaos in physical devices [2] with tiny perturbations using control methods that are based on the fundamental technique proposed by Ott, Grebogi, and Yorke [4]. A recent Army Research Laboratory (ARL) technical report [5] discusses the use of tiny-perturbation control for the transmission of information using chaotic dynamics. The information is embedded in the symbolic dynamics of the controlled chaotic trajectory. (The original idea for controlling chaos using tiny perturbations involved the stabilization of the *periodic* orbits that are embedded in the attractor of a chaotic dynamical system.)

In this report I develop a technique for controlling the symbolic dynamics [6] of chaos for the synthesis of time-limited waveforms (pulses) and for the synthesis of power spectra with desired properties. This technique is intended for the formation of pulses in high-power devices using tiny microelectronic control circuitry, so the power generation device is extremely simple and efficient, while the control device is compact, low-powered, and fast.

The technique for forming pulses, like the technique for information transmission, relies on the concept of controlling the symbolic dynamics of a chaotic system. (One can view the control of periodic orbits as control of orbits with cyclic symbolic dynamics, the transmission of information as control of orbits with aperiodic and stationary symbolic dynamics, and pulse formation as control of orbits with transient symbolic dynamics.)

In this report the concept of spectral shaping in a chaotic dynamical system is also introduced. The obtainable spectrum is constrained by the dynamics of the system (the tiny-perturbation control does not alter the basic topological structure of the attractor), and is related to pulse formation. It is possible to alter the spectrum of a chaotic system with ergodic and stationary dynamics, but in this report I limit the discussion of spectral shaping to pulse waveforms. Both ideas introduced here therefore concern the use of chaos to generate a signal class that has not been addressed previously in the context of chaos control: the transient time-limited signal. In addition to the time sequence pulses, and the spectra, I also include graphs of sinusoidal waveforms that have an amplitude modulation described by a linearly interpolated continuous-time version of the pulse waveforms. These graphs are intended to illustrate the appearance of intensity (amplitude) modulation

chaos and controlled chaotic pulses of intensity modulation. Microwave sources and lasers, among other devices, are known to exhibit this type of chaos. (The band-limited chaos that appears in these devices is more likely to be both phase and amplitude modulation.)

Because different dynamical systems have attractors with different topological structures in state space, the exact description of a chaotic system depends strongly on the specific system under study. Useful descriptions of chaotic dynamical systems for control almost without exception rely on the extraction from measured data of a simple return map (a discrete-time description of the dynamics) for a Poincaré surface of section in state space. I therefore limit the discussion here to a specific discrete-time dynamical system—the shift map [7]. (The relationship between a symbolic shift and a physical system has already been addressed, and an experiment is planned to demonstrate information transmission using this relationship [5]. This report therefore concentrates on the symbolic description. The shift map itself approximately describes the Lorenz [8] three-dimensional continuous-time dynamical system for the commonly used parameter values, and its basic properties appear in many other systems.)

The so-called double-scroll electrical oscillator that we have constructed in our laboratory, although a nonhyperbolic system of two collided Rossler attractors, can be described by one-dimensional mapping that is similar to the shift in the ways that are important for this discussion. More complex systems require the derivation of a symbolic system description from measurements; again, this technique has already been addressed [5]. More complex systems, including the double-scroll system, should provide even more flexibility in waveform synthesis.

2. Background

The shift map dynamical system [7] is described by the equation $x_{n+1} = 2x_n$ mod 1. It can therefore be described as follows in terms of a binary symbolic dynamics. If the current state of the map is $x_n = 0.b_1b_2b_3...$ in a binary fraction decimal representation,* then the shift is alternatively described by a shift-and-truncate operation on the binary fraction representation of the state point. To obtain x_{n+1}, simply shift the bits in the binary fraction for x_n to the left by one place, and discard the bit that ends up to the left of the decimal point. Thus, $x_{n+1} = 0.b_2b_3b_4...$. The equation $x_{n+1} = 2x_n \bmod 1$ can thus be replaced by the alternative model of a simple bit shift operating on an infinite binary string. (The relation between a symbolic dynamical system and a physical electrical oscillator dynamical system has been demonstrated in the context of information transmission [5], so I concentrate on the symbolic system here.)

This alternative symbolic description is appealing both from an intuitive and an analytical standpoint: It is easy to visualize and easy to construct initial points that yield a desired dynamics. If the system state point falls in the interval [0,1/2), the system is said to generate a binary symbol 0, and if the state point falls in [1/2,1], the system is said to generate a 1. Now the first bit in the binary fraction for the state point determines which of these intervals the state point falls in, and the action of the shift is to move all the bits to the left one space for each iteration of the map. The symbol sequence generated by this map, in binary fraction notation, is therefore precisely the same as the binary fraction representation of the system state point. Thus, denoting the system *symbolic state* by r, the symbolic state is related to the state space state by the identity function $r(x) = x$. One can therefore specify the symbolic dynamics by simply setting the bits in the binary fraction for the initial point to the desired symbolic dynamics.

In other systems, however, the system state and the symbolic state will be related by a function other than the identity. (In the Lorenz system, for example, $r(x)$ is practically continuous and monotonic, but is not the identity. It can, however, be transformed into the identity using a continuous monotonic coordinate change.) The problem of specifying a time-limited waveform is now reduced to the task of specifying the bits in the binary fraction for the initial point x_0. This conceptual abstraction is justified in view of the fact that with tiny controls the symbol sequence that evolves from the system can be viewed as emerging from the bits that are altered below the threshold of observability.

**Briefly, if the i^{th} place behind the decimal point is given by the binary digit (bit) b_i, then the state point of the map is $x = \sum_{i=1}^{\infty} b_i 2^{-i}$. Each real number is thus identified with an infinite binary string, and vice versa.*

First, for completeness and comparison, I briefly describe the natural dynamics and spectrum of the shift map. The natural invariant probability density for the shift [7] is simply the uniform density on the interval. The correlation function for the shift map signal is computed using the formula

$$C(m) = \int_0^1 x p(x) f^m(x)\, dx - E[x]^2,$$

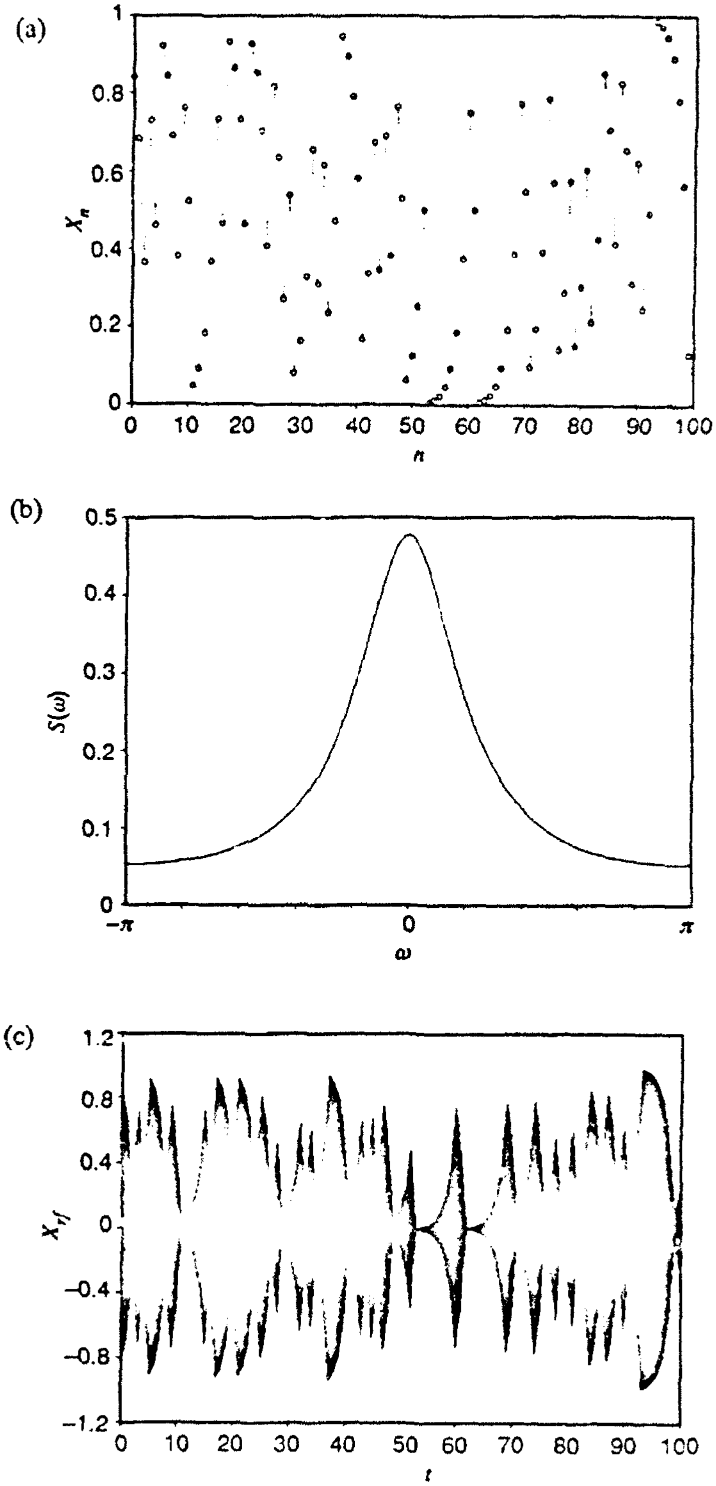
where $f^m(x)$ is the m^{th} iterate of the map, and $E[x]^2 = 1/4$ is the squared mean value of the signal. The correlation function is computed to be $C(m) = (1/12)2^{-|m|}$. (The correlation function evaluated for $m = 0$ is equal to 1/12, the variance of a uniformly distributed random variable on the unit interval.) The unnormalized power spectral density is the Fourier transform of the autocorrelation function,

$$S(\omega) = \sum_{m=-\infty}^{\infty} C(m) e^{-i\omega m}.$$

The unnormalized power spectral density is given by $S(\omega) = (1/4)/(5 - 4 \cos \omega)$. This power spectral density, because it is for a discrete-time signal (a sequence of numbers), is periodic with period 2π. Note that the power density for this signal, because it is not delta correlated, is more band limited than that of a uniform random variable, which has a flat spectrum because it is delta correlated. Figure 1(a) shows a time sequence of 100 points generated by the natural dynamics of the shift. The power spectral density, normalized so that the integrated spectral power is unity, is shown in figure 1(b). A waveform with amplitude modulation described by this time sequence with a direct linear interpolation between sequence points is shown in figure 1(c).

The waveform in figure 1(c) is described by $x_{rf}(t) = x(t) \sin (10 \pi t)$, so that the frequency of the modulated waveform is five times the sample frequency. The baseband time waveform, $x(t)$, is obtained using a linear interpolation between sample points. The use of waveforms such as sinc functions or Nyquist pulses has been avoided because these bases derive from linear signal theory [9]. In general, the best basis to use to reconstruct a time-sampled chaotic trajectory is precisely the curve that projects the trajectory from the current sample point to the next one on the surface of section. In this case, the system does not refer to a continuous time system, so the choice of basis is arbitrary, and I have used a particularly simple one.

Figure 1. (a) 100-point sequence generated by natural dynamics of map $x_{n+1} = 2x_n \bmod 1$, (b) power spectral density, and (c) modulation waveform.



3. Forming a Pulse

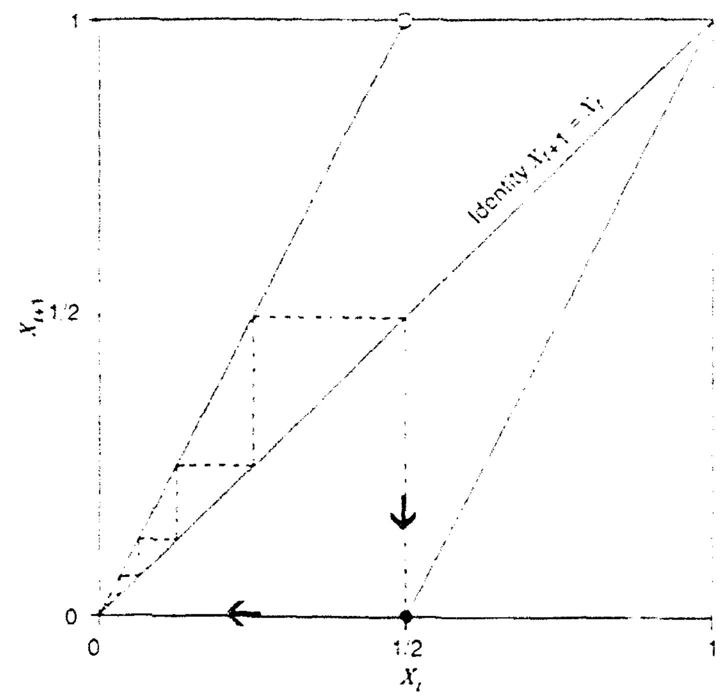
Because we can describe any allowed dynamical motion of this system by specifying the bits in the binary fraction for an initial point (and thus the symbolic future of the system), I consider pulses of the following form. Assume that the system is first in the off state or approximately so. That is, the initial point is $x_0 = 0.000 \ldots 000b_N b_{N+1} b_{N+2} \ldots$, a long string of zeros followed by a bit sequence to be specified. If the initial string of zeros is long enough, then the system can be considered to be locked to the off state by an extremely tiny control that simply kicks the system back toward zero as the shift dynamics tries to move it away. This would correspond, in a physical system or device, to locking the system in a static but unstable equilibrium state, called an unstable fixed point. I also consider that the system eventually is returned, after emission of the pulse, to the off state. Thus, the most general form for a pulse of this type is specified by the initial point (or symbol sequence) $x_0 = 0.0^N P^T 0^\infty$, where 0^N denotes a precursor string of zeros of length N, 0^∞ represents an infinite tail string of zeros, and P^T is an arbitrary string of ones and zeros of integer length T representing the pulse bits. I assume that the precursor string 0^N is arbitrarily long for computational purposes, so that the system is very close to the off state before the pulse is initiated.

The problem now is simply to specify the pulse string P^T. The simplest pulse that can exist occurs when $P^T = 1$; that is, the pulse string is one bit long and is given by the binary digit 1. I will call this the *unit pulse*, for obvious reasons, but it should not be confused with the unit amplitude impulse in linear signal theory [9]. This unit pulse, as will be shown, is a growing exponential. It is straightforward to describe the pulse amplitude as a function of time (in this case the *integer** index t): Because the single pulse bit is at first deep in the number x_0, the system state is initially off (practically). As the long string of zeros in the precursor, with the one bit deep inside, get shifted out (to the left), the system state begins to move away from off. Equivalently, the signal amplitude builds from zero and grows exponentially. The shifting of this one bit from deep inside the number to successively higher significant bit slots means that the pulse signal as a function of time is given by $x_t = 1/2\ 2^t$, $t \le 0$; and $x_t = 0$, $t > 0$.

If I define the step function s_t to be unity for $t \le 0$, and zero for $t > 0$, the unit pulse can be written as $u_t = 2^{t-1} s_t$. The pulse amplitude thus builds up exponentially with time, and then cuts off abruptly at $t = 0$. This dynamics is portrayed graphically in figure 2, which shows the graph of the shift map

**I will use the variable t to represent the discrete-time index, instead of a commonly used integer variable like n. I will also use the terminology waveform when referring to the sequence generated by the shift map. Both of these unconventional usages were chosen to avoid referring to the sequence generated by the shift as a signal, which has the connotation of information transmission and signal processing. This note deals with pulse generation, possibly for transmission from a high-powered microwave source or laser, but not intended for information transmission*

Figure 2. Unit pulse dynamics in map plane.



along with a map plane (function space) trajectory of the system. The dynamics of the state point is represented by the dynamics of a point in the map plane that successively travels between the map function and the identity line $x_{t+1} = x_t$. The reflection of the point off the map thus denotes the insertion of the value of x_t into the map. The reflection of the point off the identity line represents the setting of the value of x_t to the value of x_{t+1} from the last iterate. The system point first exits the off state along the dotted line, and the amplitude builds exponentially until it hits the point x_t = 1/2 and locks back to the off state. Note that the system locks precisely to off because all the bits following the pulse bit are zero.

The spectrum of this pulse is easily computed:

$$U(\omega) = \sum_{t=-\infty}^{t=\infty} u_t e^{-it\omega},$$

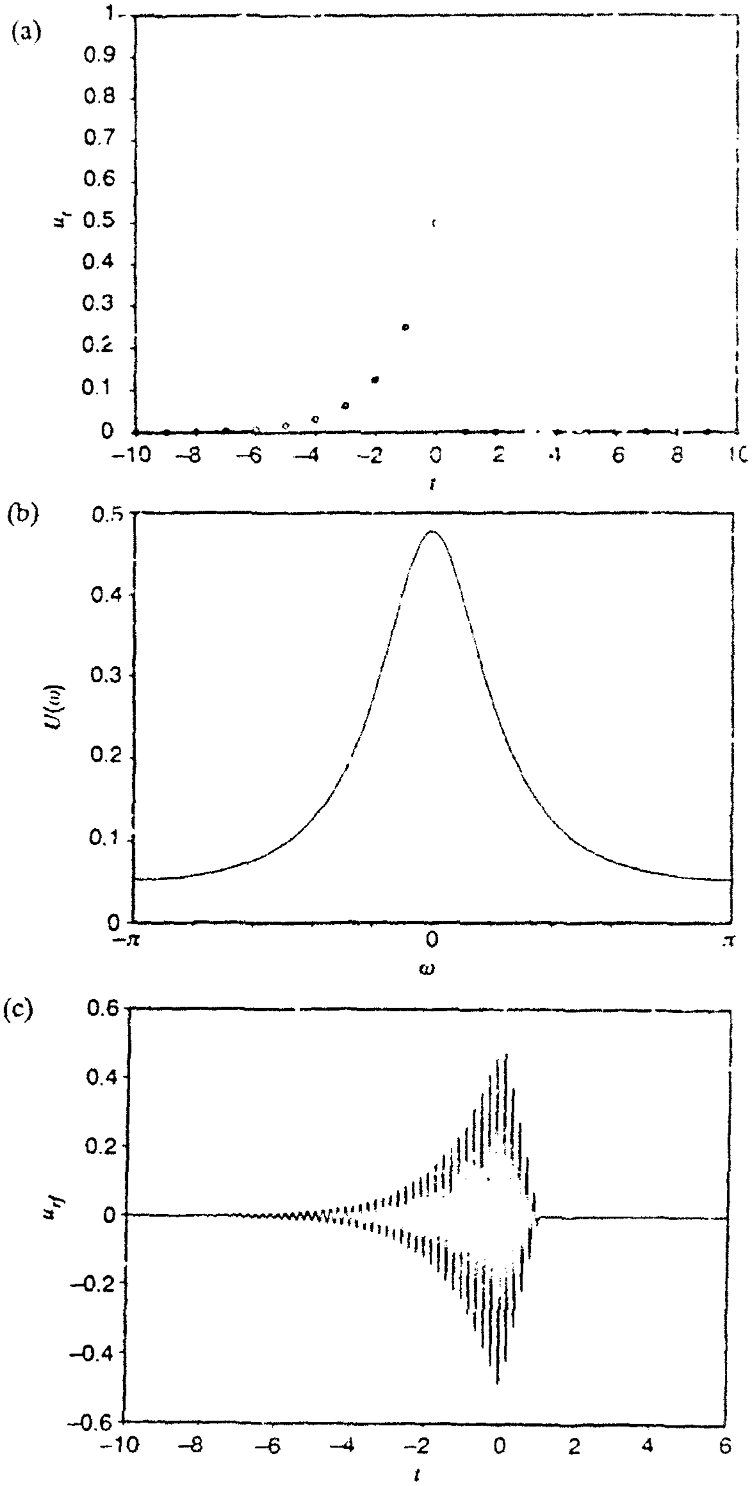
so that

$$U(\omega) = \sum_{-\infty}^{0} 2^{t-1} e^{-it\omega} = 1/2 \sum_{0}^{\infty} \left(1/2e^{i\omega}\right)^t$$

and using the closed form expression for the power series yields $U(\omega) = 1/(2 - e^{i\omega})$. The power spectrum is thus given by $|U(\omega)|^2 = 1/(5 - 4\cos\omega)$.

The unit pulse waveform and its normalized power spectrum are shown in figures 3(a) and (b), respectively. Note that the power spectrum of this time-limited pulse is the same as the power spectral density for the station-

Figure 3. Unit pulse (a) time sequence, (b) power spectrum, and (c) modulation waveform.



ary ergodic signal produced by the natural dynamics of the system. This equivalence provides a good example that two signals need not be the same to yield the same power spectrum, and shows a deeper mathematical connection between the natural dynamics and the unit pulse. (A natural trajectory of the system can be described as a random superposition of unit pulses; thus the power spectra can be shown to be the same.) An amplitude modulation waveform generated by this pulse is shown in figure 3(c), given by $u_{rf}(t) = u(t) \sin(10\,\pi t)$, where $u(t)$ is the linearly interpolated unit pulse.

More complex pulses can be formed similarly by specifying the pulse string P^T. Some pulses of length T with different pulse strings are equivalent. This occurs because any zeros at the beginning or end of the pulse string do nothing but shift the time of the pulse, an inconsequential detail in view of the fact that the precursor string is arbitrarily long. Therefore, I adopt the convention that all pulse strings must begin and end in a one. All bits between the first and last one are then arbitrary, and the number of unique pulses of length T is given by $N_T = 2^{T-2}$. Some of these are symbolic time-reversed copies of another, but the state point dynamics differ because of the different numerical significance of the bit positions.

4. Spectral Shaping

More complex pulses can be described in the context of spectral shaping. (There are other ways to differentiate between the characteristics of the pulses, but power spectra have become entrenched as a central concept in linear signal theory.) The power spectrum also has the appeal of telling what spectral bandwidth is needed for transmission of a given amount of pulse energy. It is obvious, at least, that the symbolic description is too detailed for differentiating global properties of pulses. For a long pulse length, T, there are many pulses that are nearly the same in the sense that they appear quasirandom within the pulsewidth.

During the buildup phase of a pulse, the amplitude grows exponentially. Because the pulse string is sequentially amplified by a factor of two, if I let the real number $A_0 = 0.P^T$, then the pulse amplitude as a function of time, during the buildup phase, is given by $A(t) = A_0 2^t, t \leq 0$. At $t = 0$, the leftmost bit of the pulse string is in the most significant bit (msb) slot of the binary fraction (the 1/2 place). For $t > 0$, the truncation caused by the modulo operation causes the pulse amplitude to change in a chaotic fashion (at least for most pulse strings) until the last bit (required to be a 1) is in the msb slot. Then, the pulse amplitude is 1/2, and after the next iteration, the system again returns abruptly to the off state.

Now let $p_t = \sum_{k=0}^{k=T-1} a_k u_t^k$, where $u_t^k = 2^{t-k-1} s_{t-k}$ is a unit pulse that reaches its maximum value of 1/2 at $t = k$, and a_k is either zero or one depending on the corresponding bit in the desired pulse string. Following this procedure, any of the 2^{T-2} possible pulses can be constructed from the unit pulses, in a manner roughly analogous to the formation of arbitrary pulses in linear signal theory from the unit impulse.* In this case, however, it is important to remember that the superposition is occurring bitwise in the binary fraction symbolic description of the system state, and that the linear superposition always adds bits that are never both ones. Thus, in the language of arithmetic, no carries ever happen. If one tries to use superposition of two pulses that cause a carry, the linear superposition breaks down.

The possibility of using linear superposition like this helps immensely for the problem of spectral shaping. Because all complex pulses are formed from unit pulses, the spectrum of an arbitrary pulse can be computed easily.

It is interesting that the unit impulse from classical signal theory is obtained in the limit of a large symbol alphabet from the concept introduced here of linear superposition of symbol strings. If a symbol sequence $a_1^{m_1} a_2^{m_2} a_3^{m_3} \ldots$ from a symbol alphabet of cardinality M is represented in its M-ary fraction representation, then the msb slot containing m_1 is M times more significant than the next most significant slot containing m_2. Thus, as M becomes very large, the amplitude of the shift, which is given by $0.m_1m_2m_3\ldots$ remains very close to zero if $m_1 = 0$, and is close to one if $m_1 = M - 1$. The signal amplitude is thus largely determined by the msb slot of the shift, and the unit pulse for the M-ary alphabet is a sharply rising exponential that becomes the classical unit impulse as $M \to \infty$.

Thus

$$P(\omega) = \sum_{t=-\infty}^{t=\infty} p_t e^{-i\omega t} = \sum_{t=-\infty}^{t=\infty} \left(\sum_{k=0}^{k=T-1} a_k u_t^k \right) e^{-i\omega t}.$$

Using the expression for the unit pulse, and exchanging the order of the sum yields

$$P(\omega) = \sum_{k=0}^{k=T-1} \tfrac{1}{2} a_k \sum_{t=-\infty}^{t=\infty} 2^{t-k} s_{t-k} e^{-i\omega t}.$$

The sum over time is just the Fourier transform of the unit pulse phase-shifted because the peak does not occur at $t = 0$. Thus the spectrum becomes

$$P(\omega) = \sum_{k=0}^{k=T-1} a_k \left(\frac{e^{-i\omega k}}{2 - e^{i\omega}} \right).$$

The expression for the power spectrum can be computed by noting that the squared magnitude of the amplitude spectrum is equivalent to the matrix expression

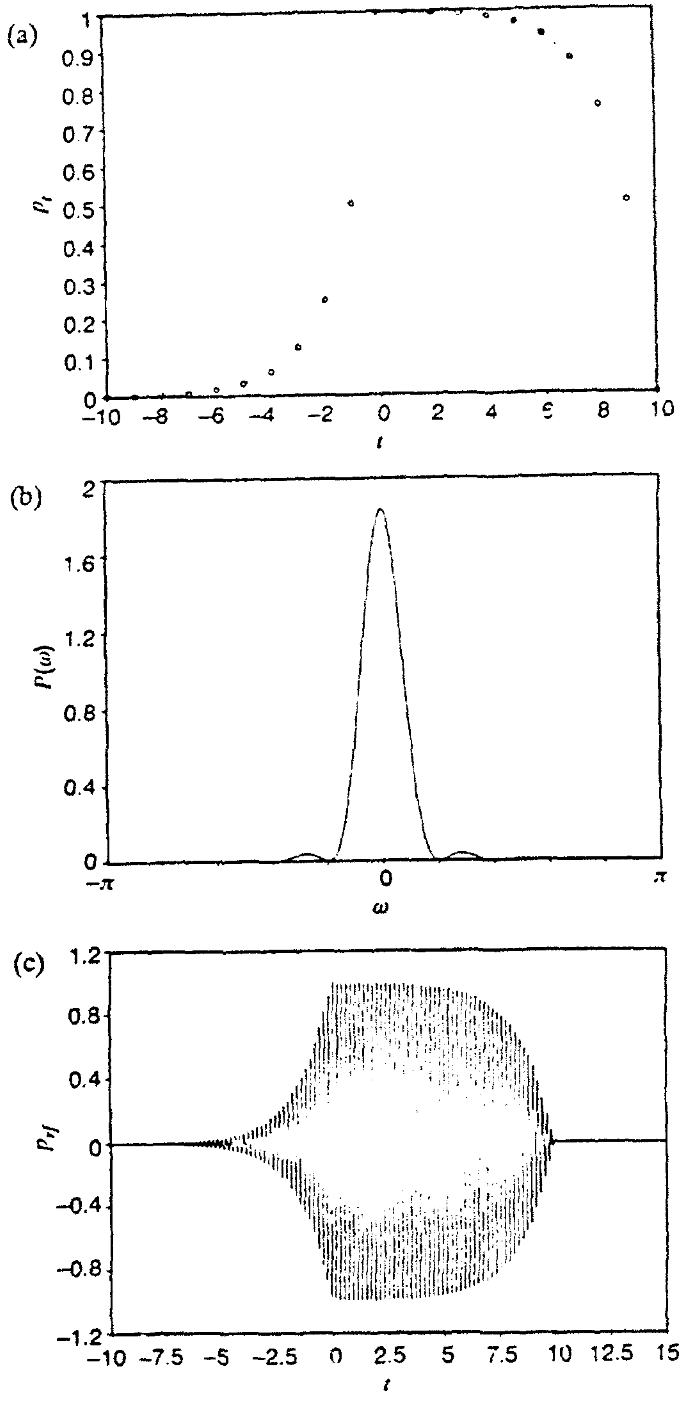
$$|P(\omega)|^2 = \frac{\mathbf{a}^T \boldsymbol{\Phi} \mathbf{a}}{5 - 4\cos(\omega)} \quad ,$$

where $\boldsymbol{\Phi} = [\phi_{kl}] = \mathbf{FF}^*$ is the matrix of products of phase terms $\phi_{kl} = e^{-i\omega(k-l)}$, $\mathbf{F} = [f_k]$ is a column vector of phase terms $f_k = e^{-i\omega k}$, and $\mathbf{a} = [a_k]$ is the column vector of unit pulse coefficients.

This expression for the power spectrum yields an elegant solution to the problem of spectral shaping. Because any of the a_k can be set to zero or one as desired, the spectrum can be varied over a large range of possible spectra. Noting that the terms in the numerator of $|P(\omega)|^2$ are the components in a Fourier synthesis of a function, then the problem reduces to the determination of the Fourier coefficients that will yield a spectrum close to the one desired. Recall that even the coefficients that are variable can only be zero or one, so this is not exactly like a classical problem of undetermined coefficients.

Instead of solving the problem in detail (which would obscure the basic idea), I will give some specific solutions that alter the spectrum from the one occurring with the natural dynamics. A simple pulse waveform constructed from the unit pulse is given by $P^{10} = 1111111111$, which is a pulse of 10 unit pulses in a row. This pulse waveform, along with its power spectrum and corresponding amplitude-modulated waveform, is shown in figure 4. The power density has now become concentrated near $\omega = 0$. The pulse therefore requires much less spectral bandwidth for transmission than either the unit pulse or the signal produced by the natural dynamics.

Figure 4. P^{10} = 1111111111 pulse (a) sequence, (b) power spectrum, and (c) modulation waveform.



Other pulse waveforms can be constructed similarly. Figure 5 shows the pulse given by P^9 = 1010101010, along with its power spectrum and amplitude modulation waveform. (This is a T = 9 pulse because the last bit is zero.) Note that the power spectrum of the pulse now has significant power density near $\omega = \pm\pi$. This is caused by the presence of the frequency component near a period two orbit of the map. Finally, figure 6 shows the pulse P^{10} = 1100110011, along with its power spectrum and amplitude modulation waveform. This pulse comes close to a period four orbit, and the spectrum is enhanced near $\omega = \pm\pi/2$. The bit patterns indicated by these pulses, if continued indefinitely, would cause the system to become locked in a periodic orbit, period two and four, respectively. With very long pulses, the corresponding Fourier component would become prominent, and would approach a delta function in the frequency domain. The large lobe at $\omega = 0$ visible in all the pulse spectra in this report would also approach a delta function. This lobe, representing the positive offset component of all the pulses, would correspond to the dc component of the signal for long pulses. Thus, one can think of the other spectral lines as representing the amplitude modulation on the dc component of the signal.

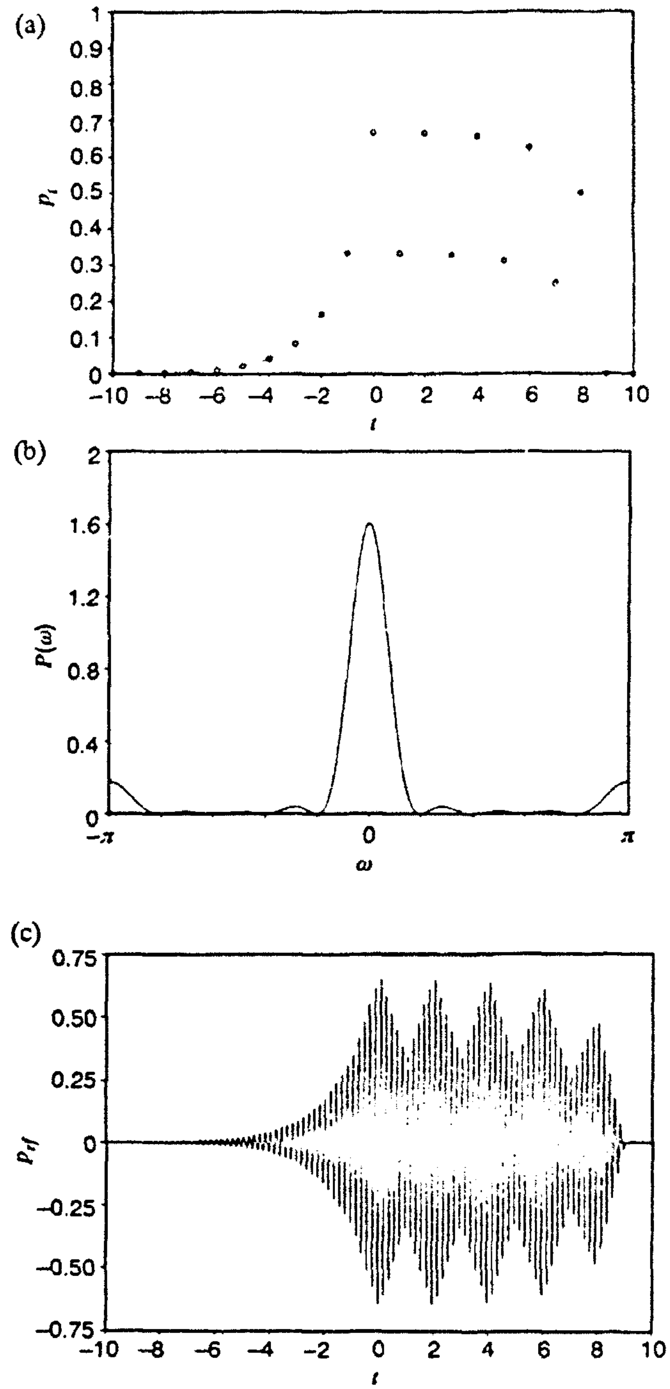
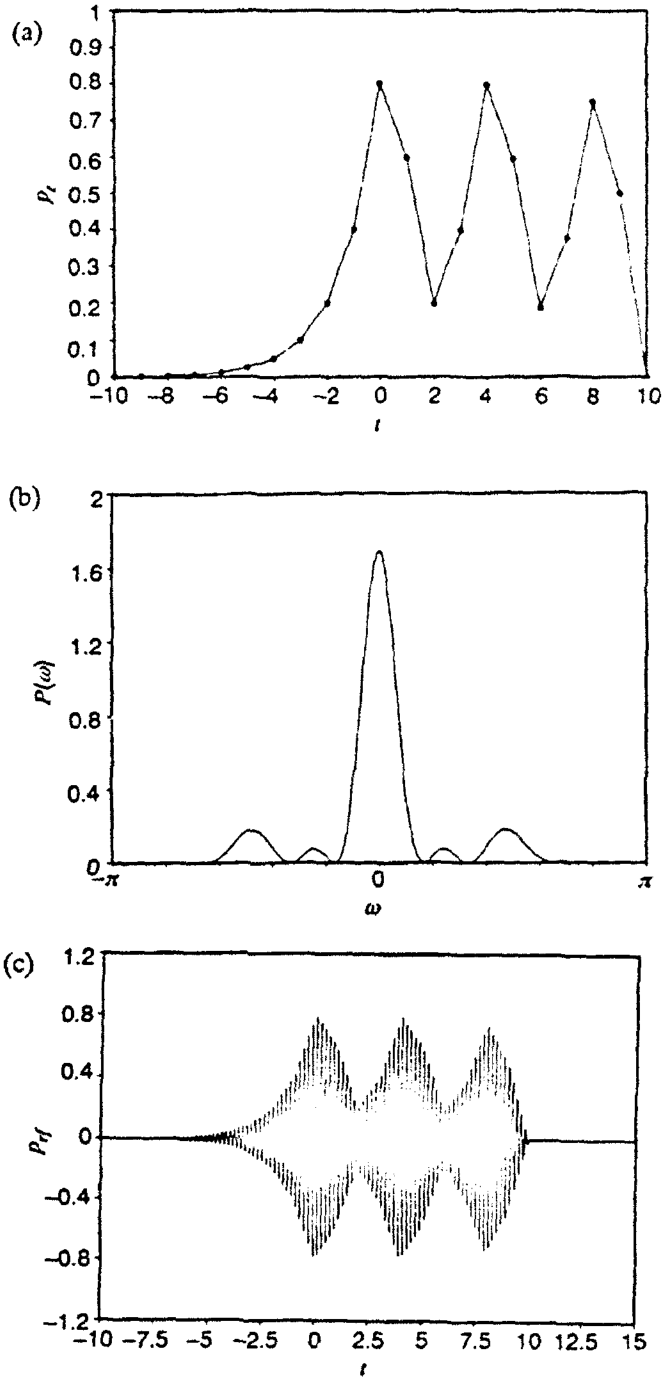


Figure 5. P^{10} = 1010101010 pulse (a) sequence, (b) power spectrum, and (c) modulation waveform.

Figure 6. P^{10} = 1100110011 pulse (a) sequence, (b) power spectrum, and (c) modulation waveform.



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